

# Note on Maximal Bisection above Tight Lower Bound

Gregory Gutin and Anders Yeo

Department of Computer Science  
Royal Holloway, University of London  
Egham, Surrey TW20 0EX, UK  
gutin|anders@cs.rhul.ac.uk

## Abstract

In a graph  $G = (V, E)$ , a bisection  $(X, Y)$  is a partition of  $V$  into sets  $X$  and  $Y$  such that  $|X| \leq |Y| \leq |X| + 1$ . The size of  $(X, Y)$  is the number of edges between  $X$  and  $Y$ . In the Max Bisection problem we are given a graph  $G = (V, E)$  and are required to find a bisection of maximum size. It is not hard to see that  $\lceil |E|/2 \rceil$  is a tight lower bound on the maximum size of a bisection of  $G$ .

We study parameterized complexity of the following parameterized problem called Max Bisection above Tight Lower Bound (Max-Bisec-ATLB): decide whether a graph  $G = (V, E)$  has a bisection of size at least  $\lceil |E|/2 \rceil + k$ , where  $k$  is the parameter. We show that this parameterized problem has a kernel with  $O(k^2)$  vertices and  $O(k^3)$  edges, i.e., every instance of Max-Bisec-ATLB is equivalent to an instance of Max-Bisec-ATLB on a graph with at most  $O(k^2)$  vertices and  $O(k^3)$  edges.

## 1 Introduction

In a graph  $G = (V, E)$ , a *bisection*  $(X, Y)$  is a partition of  $V$  into sets  $X$  and  $Y$  such that  $|X| \leq |Y| \leq |X| + 1$ . The *size* of  $(X, Y)$  is the number of edges between  $X$  and  $Y$ . In the MAX BISECTION problem we are given a graph  $G = (V, E)$  and are required to find a bisection of maximum size. Corollary 1 in the next section shows that  $\lceil m/2 \rceil$  is a tight lower bound on the maximum size of a bisection of  $G$ , where  $m = |E|$ . In what follows, for any pair  $U, W$  of disjoint sets of  $V$ ,  $(U, W)$  will denote the set of edges between  $U$  and  $W$ , and  $n$  and  $m$  will stand for the number of vertices and edges, respectively, in the graph  $G$  under consideration. In the rest of the paper,  $n$  is assumed to be even as if  $n$  is odd, we may add an isolated vertex to  $G$  without changing the maximum size of a bisection or our lower bound of  $\lceil m/2 \rceil$ .

The standard parametrization of MAX BISECTION is to decide whether  $G$  has a bisection of size at least  $k$ . (We give basic definitions on parameterized complexity later in this section.) Using the  $\lceil m/2 \rceil$  lower bound, it is easy to see that the standard parametrization of MAX BISECTION has a kernel with at most  $2k$  edges. Indeed, if  $\lceil m/2 \rceil \geq k$  the answer is YES and, otherwise,  $m \leq 2k$ . At the first glance, it looks like the size  $2k$  of this kernel is small, but it is not true. Indeed, for  $k > (m + 1)/2$ , we

have  $2k > m + 1$ , which means that the kernel is of little value from both theoretical and practical points of view.

Similar examples were given by Mahajan et al. [20] who indicated that only parameterizations above tight lower bounds or below tight upper bounds are of interest. Several results on problems parameterized above tight lower bounds have already been obtained in the literature (e.g., [1, 3, 6, 7, 12, 13, 14, 15, 16, 19, 20, 22]), but almost all results on the topic in the last couple of years were on constraint satisfaction rather than graph theoretical problems.

In this paper, we turn to graph theoretical problems parameterized above tight lower bounds and consider the following MAX BISECTION ABOVE TIGHT LOWER BOUND (MAX-BISEC-ATLB) problem: decide whether a graph  $G$  has a bisection of size at least  $\lceil m/2 \rceil + k$ , where  $k$  is the parameter. We prove that this parameterized problem has a kernel with  $O(k^2)$  vertices and  $O(k^3)$  edges. Thus, in particular, MAX-BISEC-ATLB is fixed-parameter tractable. A closely related result to ours is by Bollobás and Scott [3] who proved that the problem of deciding whether a graph  $G$  has a maximum cut of size at least  $\frac{m}{2} + \sqrt{\frac{m}{8}} + k$ , where  $k$  is the parameter, has an algorithm of running time  $O(2^{O(k^4)} + n + m)$ , i.e., the problem is fixed-parameter tractable. Note that the problem considered by Bollobás and Scott [3] is parameterized above a tight lower bound as  $\lceil \frac{m}{2} + \sqrt{\frac{m}{8}} + \frac{1}{64} - \frac{1}{8} \rceil$  is a tight lower bound on the maximum size of a cut, which was first proved by Edwards [9].

A *parameterized problem* is a subset  $L \subseteq \Sigma^* \times \mathbb{N}$  over a finite alphabet  $\Sigma$ .  $L$  is *fixed-parameter tractable* if the membership of an instance  $(x, k)$  in  $\Sigma^* \times \mathbb{N}$  can be decided in time  $f(k)|x|^{O(1)}$ , where  $f$  is a computable function of the parameter  $k$ . If the nonparameterized version of  $L$  (where  $k$  is just a part of the input) is NP-hard, then the function  $f(k)$  must be superpolynomial provided  $P \neq NP$ . Often  $f(k)$  is “moderately exponential,” which makes the problem practically feasible for small values of  $k$ . Thus, it is important to parameterize a problem in such a way that the instances with small values of  $k$  are of real interest.

Given a parameterized problem  $L$ , a *kernelization* of  $L$  is a polynomial-time algorithm that maps an instance  $(x, k)$  to an instance  $(x', k')$  (the *kernel*) such that (i)  $(x, k) \in L$  if and only if  $(x', k') \in L$ , (ii)  $k' \leq g(k)$ , and (iii)  $|x'| \leq h(k)$  for some functions  $g$  and  $h$ . The function  $h(k)$  is called the *size* of the kernel.

It is well-known that a parameterized problem  $L$  is fixed-parameter tractable if and only if it is decidable and admits a kernelization. Due to applications, low degree polynomial size kernels are of main interest. Unfortunately, many fixed-parameter tractable problems do not have kernels of polynomial size unless the polynomial hierarchy collapses to the third level [4, 5, 10]. For further background and terminology on parameterized complexity we refer the reader to the monographs [8, 11, 21].

## 2 Results

A result similar to the following lemma but for cuts rather than bisections was apparently first proved by Haglin and Venkatesan [17].

**Lemma 1.** *If  $M$  is a matching in a graph  $G$ , then  $G$  has a bisection of size at least  $\lceil m/2 \rceil + \lfloor |M|/2 \rfloor$ .*

*Proof.* Recall that we may assume that  $n$  is even and let  $p = n/2$ . Let  $U = u_1, u_2, \dots, u_p$  and  $V = v_1, v_2, \dots, v_p$  be two disjoint sequences of vertices of  $G$  such that  $M = \{u_1v_1, \dots, u_{|M|}v_{|M|}\}$ . Starting from empty sets  $X$  and  $Y$ , for each  $i = 1, 2, \dots, p$ , place  $u_i$  in  $X$  or  $Y$  with probability  $1/2$  and place  $v_i$  in the other set. Observe that the expectation of the size of the bisection is  $|M| + (m - |M|)/2$  since the probability of each edge of  $M$  to be between  $X$  and  $Y$  is  $1$  and the probability of any other edge to be between  $X$  and  $Y$  is  $1/2$ . Thus, there is a bisection in  $G$  of size at least  $\lceil m/2 + |M|/2 \rceil \geq \lceil m/2 \rceil + \lfloor |M|/2 \rfloor$ .

We can find such a bisection by derandomizing the above randomized procedure using the well-known method of conditional probabilities, see, e.g., Chapter 15 in [2] or Chapter 26 in [18]. This derandomization leads to a greedy algorithm in which at Step  $i$  ( $1 \leq i \leq p$ ) we place  $u_i$  in  $X$  and  $v_i$  in  $Y$  rather than the other way around if and only if  $|u_i, Y| + |v_i, X| \geq |u_i, X| + |v_i, Y|$ , where  $X$  and  $Y$  are sets constructed before Step  $i$  (here  $u_i$  stands for  $\{u_i\}$ , etc.). The greedy algorithm takes time  $O(m + n)$ .  $\square$

**Corollary 1.** *A graph  $G$  has a bisection of size at least  $\lceil m/2 \rceil$  and this lower bound on the maximum size of a bisection is tight.*

*Proof.* The first part of the claim follows immediately from Lemma 1. To see that  $\lceil m/2 \rceil$  is tight, it suffices to consider the star  $K_{1,m}$  for any odd  $m$ .  $\square$

**Theorem 1.** *The problem MAX-BISEC-ATLB has a kernel with  $O(k^2)$  vertices and  $O(k^3)$  edges.*

*Proof.* Recall that we may assume that  $n$  is even, as otherwise we can add an isolated vertex. Let  $M$  be a maximal matching in a graph  $G = (V, E)$ . Such a matching can be found in time  $O(n + m)$ . If  $|M| \geq 2k$ , then by Lemma 1, the answer to MAX-BISEC-ATLB is YES. Thus, assume that  $|M| < 2k$ . For each vertex  $x$  covered by  $M$ , let  $S(x)$  be the smallest of the following two sets:  $N(x) \setminus V(M)$  and  $V \setminus (V(M) \cup N(x))$ , where  $N(x)$  is the set of neighbors of  $x$  and  $V(M)$  is the set of vertices covered by  $M$ . Now consider two cases.

**Case 1:** There is a vertex  $z \in V(M)$  with  $|S(z)| \geq 2k - (|M| - 1)$ . Let  $X'$  be any set of size  $2k - (|M| - 1)$  in  $N(z) \setminus V(M)$  and let  $Y'$  contain  $z$  and  $2k - (|M| - 1) - 1$  vertices from  $V(G) \setminus (V(M) \cup N(z))$ . Note that  $|X'| = |Y'| = 2k - |M| + 1$  and there are  $2k - |M| + 1$  edges between  $X'$  and  $Y'$ . Furthermore  $X'$  and  $Y'$  are independent sets of vertices. Set  $X = X'$  and  $Y = Y'$ , and let  $M'$  be the set of edges in  $M$  minus the edge incident to  $z$ . For each edge  $uv$  in  $M'$ , place  $u$  in  $X$  or  $Y$  with probability  $1/2$  and place  $v$  in the other set. Partition the vertices of  $G$  still not in  $X \cup Y$  into pairs and use the randomized procedure of Lemma 1 to assign those vertices to either  $X$  or  $Y$ .

Observe that the expected number of edges between  $X$  and  $Y$  equals  $|X', Y'| + |M'| + f/2$ , where  $f$  is the number of edges of  $G$  not belonging to  $(X', Y')$  or  $M'$ . Thus, the expected number of edges between  $X$  and  $Y$  is at least

$$m/2 + [(2k - |M| + 1) + (|M| - 1)]/2 = m/2 + k.$$

Similarly to Lemma 1, we can derandomize the randomized procedure from the first paragraph of this proof to obtain a greedy-type algorithm producing a bisection of

size at least  $\lceil m/2 \rceil + k$ .

**Case 2:**  $|S(x)| < 2k - |M| + 1$  for all  $x \in V(M)$ . We start by performing the following reduction: If  $G$  has an independent set  $I$  of size  $n/2 + j$  (with  $j > 0$ ) such that all vertices in  $I$  have the same neighborhood (and  $I$  is maximal with respect to the two properties), then delete  $2j$  of the vertices in  $I$  from  $G$ . We may do this reduction as any bisection of  $G$  will have at least  $j$  vertices from  $I$  in each part. Note that if the reduction is performed, the new graph  $G$  cannot have an independent set  $I$  of size  $n/2 + j$  (with  $j > 0$ ) such that all vertices in  $I$  have the same neighborhood (here  $n := n - 2j$ ).

Now we will prove that  $n = O(k^2)$ . Let  $S = \cup_{x \in V(M)} S(x)$  and note that  $|S| \leq 2|M|(2k - |M|)$ . Let  $Z = V(G) \setminus (V(M) \cup S)$  and note that  $|Z| \geq n - 2|M| - 2|M|(2k - |M|)$ . The maximum value of the function  $f(t) = 2t(2k - t + 1)$  is obtained when  $t = k + 1/2$ . However, for integral  $t$ , it is obtained when  $t = k$  or  $t = k + 1$ , which in both cases gives  $f(k) = f(k + 1) = 2k(k + 1)$ . Therefore  $|Z| \geq n - 2k(k + 1)$ . As all vertices in  $Z$  have exactly the same neighborhood, we have  $|Z| \leq n/2$ , and thus we have the following:  $n/2 \geq |Z| \geq n - 2k(k + 1)$  implying  $4k(k + 1) \geq n$ .

Hence, we have a kernel with at most  $4k(k + 1) = O(k^2)$  vertices. Recall that  $|M| < 2k$  and observe that  $V(G) \setminus V(M)$  is independent. Thus, the number of edges in the kernel is at most

$$|V(M)| \cdot |V(G) \setminus V(M)| + \binom{|V(M)|}{2} \leq 4kn + 8k^2 = O(k^3).$$

□

### 3 Open Problems

We have proved that MAX-BISEC-ATLB has a kernel with  $O(k^2)$  vertices and  $O(k^3)$  edges. It would be interesting to obtain a kernel with fewer vertices and/or edges.

We can obtain a stronger lower bound for the maximum size of a bisection in a graph  $G = (V, E)$ . Choose a random bisection  $(X, Y)$  in  $G$  by randomly choosing  $n/2$  vertices of  $G$ . Observe that the probability  $p$  of an edge being in  $(X, Y)$  is  $\frac{n}{2(n-1)}$ . Thus,  $\lceil pm \rceil$  is a lower bound on the maximum size of a bisection. Observe that this bound is tight as the extreme graphs include not only stars, but also complete graphs. It would be interesting to determine the parameterized complexity of the following problem: given a graph  $G$ , decide whether  $G$  has a bisection of size at least  $\lceil pm \rceil + k$ , where  $k$  is the parameter.

The situation between our main result and the last open problem is similar to that between the above-mentioned result of Bollobás and Scott and the following open question from [20]. Determine the parameterized complexity of the following problem: given a connected graph  $G$ , decide whether  $G$  has a cut of size at least  $\frac{m}{2} + \frac{n-1}{4} + k$ , where  $k$  is the parameter. Note that  $\frac{m}{2} + \frac{n-1}{4}$  is a tight lower bound on the maximum size of a cut of a connected graph, which was first proved by Edwards [9]. It is easy to check that  $\frac{m}{2} + \sqrt{\frac{m}{8} + \frac{1}{64}} - \frac{1}{8} \leq \frac{m}{2} + \frac{n-1}{4}$ .

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